ADDENDUM TO "FROBENIUS AND CARTIER ALGEBRAS OF STANLEY-REISNER RINGS" [J. ALGEBRA 358 (2012) 162-177]

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ABSTRACT. We give a purely combinatorial characterization of complete Stanley-Reisner rings having a principally generated (equivalently, finitely generated) Cartier algebra.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a complete local ring of prime characteristic $p > 0$. The notion of Cartier algebra, introduced by K. Schwede [6] and developed by M. Blickle [3], has received a lot of attention due to its role in the study of test ideals. More precisely, the ring of Cartier operators on R is the graded, associative, not necessarily commutative ring

$$\mathcal{C}(R) := \bigoplus_{e \geq 0} \operatorname{Hom}_R(F_*^e R, R),$$

where $F_*^e R$ denotes the ring R with the left R-module structure given by the e-th iterated Frobenius map $F^e : R \longrightarrow R$, i.e. the left R-module structure given by $r \cdot m := r^{p^e} m$. One should mention that, using Matlis duality, the Cartier algebra of R corresponds to the Frobenius algebra of the injective hull of the residue field $E_R(R/\mathfrak{m})$ introduced by G. Lyubeznik and K. E. Smith in [5].

Let $S = K[\![x_1, \ldots, x_n]\!]$ be the formal power series ring over a field K. In this note we will assume that $\operatorname{char}(K) = p > 0$. Given a simplicial complex Δ with vertex set $[n] := \{1, 2, \ldots, n\}$ one may associate a squarefree monomial ideal $I_\Delta := (\prod_{i \in F} x_i \mid F \subset [n], F \not\in \Delta)$ in S via the *Stanley-Reisner correspondence*. Building upon an example of M. Katzman [4], the first author together with A. F. Boix and S. Zarzuela [1] studied Cartier algebras of *complete Stanley-Reisner rings* $R := S/I_\Delta$ associated to Δ. One of the main results obtained in [1] is that these Cartier algebras can be either principally generated or infinitely generated as an R-algebra.

Theorem 1 ([1, Theorem 3.5])**.** *With the above notation, set $R := S/I_\Delta$. Assume that each x_i divides some minimal monomial generator of I_Δ. Then, the following are equivalent:*

(1) *The Cartier algebra $\mathcal{C}(R)$ is principally generated.*
(2) $I_\Delta^{[2]} : I_\Delta = I_\Delta^{[2]} + (\mathbf{x}^{\mathbf{1}})$.

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Otherwise the Cartier algebra $\mathcal{C}(R)$ is infinitely generated. Here $I_\Delta^{[2]}$ denotes the second Frobenius power of I_Δ and $\mathbf{x^1} := x_1x_2\cdots x_n$.

Remark 2. Set $V := \{\, i \mid x_i \text{ divides some minimal monomial generator of } I_\Delta \,\}$.

(i) The condition x_i *divides some minimal monomial generator of* I_Δ (equivalently, $V = [n]$) is only used to simplify the notations of Theorem 1. If it is not satisfied, i.e. Δ is a cone over the vertex i, then we have that $\mathcal{C}(R)$ is principally generated if and only if $I_\Delta^{[2]} : I_\Delta = I_\Delta^{[2]} + (\prod_{i\in V} x_i)$. We can always reduce to the case $V = [n]$ since there is always a simplicial complex Δ' on V such that $\Delta = \Delta' * 2^{[n]\setminus V}$ so the result follows from Lemma 3 below.

(ii) The original result in [1] has a slightly different formulation in terms of the colon ideals $I_\Delta^{[p^e]} : I_\Delta$, $e \geq 1$, but it was already noticed in [1, Remark 3.1.2] that one may reduce to the case $p = 2$ and $e = 1$. We also point out that Theorem 1 also holds in the case $\mathrm{ht}(I_\Delta) = 1$ that was treated separately in [1] for clearness.

Lemma 3. *Let Δ be a simplicial complex with vertex set $[n]$. Assume that there exists a simplicial complex Δ' on $V \subseteq [n]$ such that $\Delta = \Delta' * 2^{[n]\setminus V}$. Then, $\mathcal{C}(S/I_\Delta)$ is principally generated if and only if so does $\mathcal{C}(S/I_{\Delta'})$.*

Proof. For $S' := K[\![x_i \mid i \in V]\!]$, we have $S/I_\Delta \cong (S'/I_{\Delta'})[\![x_i \mid i \notin V]\!]$. Then the result follows from the description of the Cartier algebra in terms of the colon ideals $I_\Delta^{[2]} : I_\Delta$ (see [1] and the references therein). ☐

2. A characterization of principally generated Cartier algebras

The Cartier algebra of an F-finite complete Gorenstein local ring R is principally generated as a consequence[1] of [5, Example 3.7]. The converse holds true for F-finite normal rings (see [3]). Complete Stanley-Reisner rings are F-finite but, almost always non-normal and when discussing examples at the boundary of the Gorenstein property one can even find examples of principally generated Cartier algebras that are not even Cohen-Macaulay. The authors of [1] could not find the homological conditions that tackle this property so the aim of this note is to address this issue. Our main result is a very simple combinatorial criterion in terms of the simplicial complex Δ. To this purpose we recall that a *facet* of a simplicial complex Δ is a maximal face with respect to inclusion. We say a face $F \in \Delta$ is *subfacet* if $F \cup \{i\}$ is a facet for some $i \notin F$.

Theorem 4. *Under the same assumptions as in Theorem 1, the following are equivalent.*

(a) *The Cartier algebra $\mathcal{C}(R)$ is principally generated.*

(b) *Any subfacet of Δ is contained in at least two facets.*

Proof. For a monomial $\mathsf{m} = \prod_{i=1}^n x_i^{a_i} \in S$, set $\mathrm{supp}(\mathsf{m}) := \{\, i \mid a_i \neq 0 \,\}$ and $\mathrm{supp}_{\mathbf{2}}(\mathsf{m}) := \{\, i \mid a_i \geq 2 \,\}$. Note that $\mathsf{m} \in I_\Delta^{[2]}$ if and only if $\mathrm{supp}_{\mathbf{2}}(\mathsf{m}) \notin \Delta$. Furthermore, under the assumption that $\mathrm{supp}(\mathsf{m}) \neq [n]$, we have $\mathsf{m} \in I_\Delta^{[2]} + (\mathbf{x^1})$ if and only if $\mathrm{supp}_{\mathbf{2}}(\mathsf{m}) \notin \Delta$.

[1] Using Matlis duality.

(a) $\Rightarrow$ (b): By Theorem 1, it suffices to show that $I_\Delta^{[2]} : I_\Delta = I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$ implies (b), and the same is true for the proof of the converse implication.

Assume that Δ does not satisfy (b). Then we may assume that $\{1, 2, \ldots, l\}$ is a subfacet, and it is contained in a unique facet $\{1, 2, \ldots, l+1\}$. Set

$$\mathsf{m} := \left(\prod_{i=1}^{l} x_i^2\right) \cdot \left(\prod_{i=l+2}^{n} x_i\right).$$

Clearly, $\mathsf{m} \notin I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$. Take any monomial $\mathsf{n} \in I_\Delta$. Since $\{1, 2, \ldots, l+1\} \in \Delta$, n can be divided by x_j for some $l+2 \le j \le n$. Then $\operatorname{supp}_\mathbf{2}(\mathsf{mn}) \supseteq \{1, 2, \ldots, l, j\}$, which is not a face of Δ. It follows that $\mathsf{mn} \in I_\Delta^{[2]}$. Summing up, we have $\mathsf{m} \in (I_\Delta^{[2]} : I_\Delta) \setminus I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$. Hence the condition (a) does not hold, and we are done.

(b) $\Rightarrow$ (a): Assume that the condition (b) is satisfied. Since $I_\Delta^{[2]} : I_\Delta \supseteq I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$ always holds, it suffices to prove that $I_\Delta^{[2]} : I_\Delta \subseteq I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$, equivalently, $\mathsf{m} \notin I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$ implies $\mathsf{m} \notin I_\Delta^{[2]} : I_\Delta$. So take a monomial $\mathsf{m} \in S$ with $\mathsf{m} \notin I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$. If $\#\operatorname{supp}(\mathsf{m}) \le n-2$ and $i \notin \operatorname{supp}(\mathsf{m})$, then $x_i\mathsf{m} \notin I_\Delta^{[2]} + (\mathbf{x}^\mathbf{1})$, and $x_i\mathsf{m} \notin I_\Delta^{[2]} : I_\Delta$ implies $\mathsf{m} \notin I_\Delta^{[2]} : I_\Delta$. Hence we can replace m by $x_i\mathsf{m}$ in this case. Repeating this operation, we may assume that $\#\operatorname{supp}(\mathsf{m}) = n-1$. Let x_l be the only variable which does not divide m.

Set $F := \operatorname{supp}_\mathbf{2}(\mathsf{m})$. Since $\mathsf{m} \notin I_\Delta^{[2]}$, we have $F \in \Delta$. Moreover, there is a facet $G \in \Delta$ with $G \supseteq F$ and $l \notin G$. To see this, take any facet $H \in \Delta$ with $H \supseteq F$. If $l \notin H$, then we can take H as G. If $l \in H$, then the subfacet $H \setminus \{l\}$ is contained in a facet H' other than H by the condition (b). Clearly, we can take H' as G. Replacing m by $(\prod_{i \in G \setminus F} x_i) \cdot \mathsf{m}$, we may assume that $F = \operatorname{supp}_\mathbf{2}(\mathsf{m})$ is a facet which does not contain l. Then $\mathsf{n} := x_l \cdot \prod_{i \in F} x_i$ is contained in I_Δ, since $\operatorname{supp}(\mathsf{n}) = F \cup \{l\}$ is not a face of Δ. However, it is easy to see that $\operatorname{supp}_\mathbf{2}(\mathsf{mn}) = F \in \Delta$ and $\mathsf{mn} \notin I_\Delta^{[2]}$. It follows that $\mathsf{m} \notin I_\Delta^{[2]} : I_\Delta$. This is what we wanted to prove. $\square$

Remark 5. Under the assumption that each variable x_i divides some minimal monomial generator of I_Δ, equivalently Δ is not a cone over any vertex, one may check out that the condition on the Cartier algebra of a complete Stanley-Reisner ring $R = S/I_\Delta$ being principally generated is a topological property of the geometric realization X of Δ. In fact, by Theorem 4, $\mathcal{C}(R)$ is not principally generated if and only if there is an open subset $U \subset X$ which is homeomorphic to $\{ (x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m \ge 0 \}$ for some $m \in \mathbb{N}$. However, the condition that Δ is not a cone over any vertex is *not* topological. In this sense, being principally generated is not a topological condition. This is quite parallel to the relation between Gorenstein and Gorenstein* properties of simplicial complexes where we have that Δ is Gorenstein if and only if $\Delta = \Delta' * 2^{[n]\setminus V}$ for some Gorenstein* complex Δ' on some $V \subseteq [n]$ and Gorenstein* is a topological property (See[2] [7, §II.5]).

Despite the fact that using Theorem 4 one may construct many simplicial complexes satisfying that the Cartier algebra $\mathcal{C}(R)$ is principally generated, e.g. triangulations of

[2]The notation $\Delta = \operatorname{core}\Delta$ in [7, §II.5] corresponds to $V = [n]$ in our notation.

manifolds without boundary, it seems that there is no tight relation to any homological conditions on R. The best we can say in this direction is the following. For the definitions of *Buchsbaum* complexes* and undefined terminologies we refer to [7] and [2].

Corollary 6. *If Δ is Buchsbaum* (in particular, doubly Cohen-Macaulay, or Gorenstein*) over some field K, then $\mathcal{C}(R)$ is principally generated.*

Proof. Suppose that Δ is Buchsbaum* but $\mathcal{C}(R)$ is not principally generated. Since Δ is Buchsbaum*, Δ is not a cone over any vertex. Hence there is a subfacet σ contained in a unique maximal face τ by Theorem 4. Clearly, $\mathrm{cost}_\Delta(\sigma) = \Delta \setminus \{\sigma, \tau\}$ and $\mathrm{cost}_\Delta(\tau) = \Delta \setminus \{\tau\}$. Hence we have $H_d(\Delta, \mathrm{cost}_\Delta(\sigma); K) = 0$ and $H_d(\Delta, \mathrm{cost}_\Delta(\tau); K) = K$, and the map $H_d(\Delta, \mathrm{cost}_\Delta(\sigma); K) \longrightarrow H_d(\Delta, \mathrm{cost}_\Delta(\tau); K)$ can not be surjective. It means that Δ is not Buchsbaum* so we get a contradiction. $\square$

This result together with Lemma 3 allows us to give a direct proof of the fact that a complete Gorenstein Stanley-Reisner ring has a principally generated Cartier algebra since Δ is Gorenstein if and only if $\Delta = \Delta' * 2^{[n]\setminus V}$ for some Gorenstein* complex Δ' on some $V \subseteq [n]$.

Example 7. (i) Consider the 1-dimensional simplicial complex Δ in Figure 1 below. Δ is Cohen-Macaulay and $\mathcal{C}(R)$ is principally generated, but Δ is not doubly Cohen-Macaulay so it is not Buchsbaum* as well.

(ii) Let Δ be the simplicial complex with facets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 5\}$ and $\{2, 5\}$ (see Figure 2 below). Then, $\mathcal{C}(R)$ is principally generated but Δ is not pure.

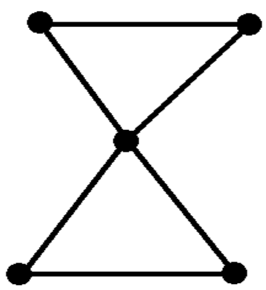

FIGURE 1

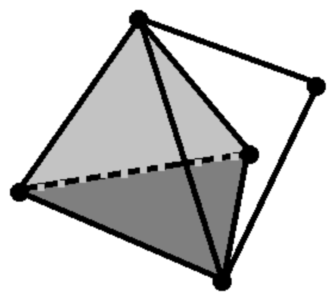

FIGURE 2

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